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A Counterexample in Linear–Quadratic Games: Existence of Nonlinear Nash Solutions[1]

T. BASAR[2]

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Abstract. Via a 2-stage linear–quadratic 2-person nonzero-sum game, we show that such decision problems might admit nonlinear Nash solutions. As compared with the linear strategy, a nonlinear Nash policy might lead to a better performance for at least one of the players.

Key Words. Game theory, linear–quadratic games, differential games, nonzero-sum games, Nash equilibria.

1. Introduction

Hitherto, it has been a common belief in control and game theory literature that 2-person deterministic decision problems with linear state dynamics, quadratic payoff functions, and with the classical information structure admit only affine Nash equilibrium solutions (see, e.g., Refs. 1–3). Attempts have been made in Refs. 1 and 2 to prove the uniqueness of affine Nash solutions for such decision problems and for general closed-loop (CL) strategy spaces. However, authors of both references are sidetracked by their assumptions of linearity and nomemory restriction on admissible strategies in their proof of uniqueness of the Nash solutions. In this paper, we will show that it is not possible to obtain a uniqueness result for dynamic nonzero-sum games, with the classical CL strategy space for at least one of the players. We actually provide a counterexample and report on the existence of a class of

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[2] Research Scientist, Marmara Research Institute, The Scientific and Technical Research Council of Turkey, Gebze-Kocaeli, Turkey.

nonlinear equilibrium solutions for a simple 2-stage linear–quadratic nonzero-sum (LQNZS) game. The example given in the paper is the simplest LQNZS game which admits nonlinear Nash solutions.

2. LQNZS Game and Derivation of Nonlinear Nash Solutions

Consider the 2-stage LQNZS game defined by the difference equations

$$x(1) = x(0) + u(0) + v(0), \qquad x(0) = x_0, \tag{1-1}$$

$$x(2) = x(1) + u(1), \tag{1-2}$$

where all variables are scalar and take their values in $\mathscr{R}^1$. $u(0)$ and $u(1)$ are control variables of Player 1 at stages zero and one, respectively, and $v(0)$ is the control variable of Player 2 at stage zero. Player 1 has access to x_0 at stage zero and both x_0 and $x(1)$ at stage one. Player 2 acts only at stage zero and has access to x_0. We denote by Γ_0 the class of all measurable maps from $\mathscr{R}^1$ onto $\mathscr{R}^1$, by Γ_1 those that map $\mathscr{R}^1 \times \mathscr{R}^1$ onto $\mathscr{R}^1$. At stage zero, Players 1 and 2 pick, respectively, $\gamma_0(\cdot) \in \Gamma_0$, $\gamma_2(\cdot) \in \Gamma_0$; and, at stage one, Player 1 picks $\gamma_1(.,.) \in \Gamma_1$. Then, the costs incurred to Players 1 and 2 are given by J_1 and J_2, respectively, where

$$J_1 = x^2(2) + u^2(1) + u^2(0), \tag{2-1}$$

$$J_2 = x^2(2) + v^2(0) + \beta u^2(1), \qquad \beta \geqslant 0, \tag{2-2}$$

with

$$u(0) = \gamma_0(x_0), \qquad u(1) = \gamma_1(x(1), x_0), \qquad v(0) = \gamma_2(x_0).$$

With these definitions, $\{\gamma_0{}^* \in \Gamma_0,\ \gamma_1{}^* \in \Gamma_1,\ \gamma_2{}^* \in \Gamma_0\}$ constitutes a Nash strategy triple to the game posed above, if it satisfies

$$J_1(\gamma_0{}^*, \gamma_1{}^*, \gamma_2{}^*) \leqslant J_1(\gamma_0, \gamma_1, \gamma_2{}^*), \tag{3-1}$$

$$J_2(\gamma_0{}^*, \gamma_1{}^*, \gamma_2{}^*) \leqslant J_2(\gamma_0{}^*, \gamma_1{}^*, \gamma_2), \tag{3-2}$$

for all $\gamma_0 \in \Gamma_0$, $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_0$.

Now, for any Nash equilibrium triple $\{u(1) = \gamma_1(x(1), x_0),\ u(0) = \gamma_0(x_0),\ v(0) = \gamma_2(x_0)\}$, the dependence of $\gamma_1(.,.)$ on $x(1)$ and x_0 will be through

$$\gamma_1(x(1), x_0) = -\tfrac{1}{2}x(1) + \psi(x(1), \bar{x}_1), \tag{4-1}$$

where

$$\bar{x}_1 = x_0 + \gamma_0(x_0) + \gamma_2(x_0), \tag{4-2}$$

and $\psi(.,.)$ is any scalar function of two variables with the property that $\psi(y, y) = 0 \; \forall \, y \in \mathscr{R}^1$. We note that, for fixed $\gamma_0(\cdot)$, $\gamma_2(\cdot)$, the strategy defined by (4–1) is *unique* in *value* (which is $-\frac{1}{2}\bar{x}_1$) but *nonunique in representation* as a control policy of Player 1 at stage one. If we now take $\psi(.,.)$ to be $\psi(y, z) = (y^2 - z^2)p$, where p is any scalar parameter, the optimal Nash strategies $\gamma_0{}^*(x_0)$, $\gamma_2{}^*(x_0)$ that correspond to this functional form will be given by (after some rather extensive manipulations, details of which are given in the Appendix)

$$\gamma_0{}^*(x_0) = [1/12p(1-\beta)][[\tfrac{3}{4}(1+\beta) + 9/2] \\ - \sqrt{\{[\tfrac{3}{4}(1+\beta) + 9/2]^2 + 36p(1-\beta)\, x_0\}}], \tag{5-1}$$

$$\gamma_2{}^*(x_0) = -x_0 - [1/4p(1-\beta)][[\tfrac{3}{4}(1+\beta) + 9/2] \\ - \sqrt{\{[\tfrac{3}{4}(1+\beta) + 9/2]^2 + 36p(1-\beta)\, x_0\}}], \tag{5-2}$$

provided that

$$p(1-\beta) \neq 0, \tag{5-3}$$

$$[\tfrac{3}{4}(1+\beta) + 9/2]^2 + 36p(1-\beta)\, x_0 \geqslant 0. \tag{5-4}$$

When $\beta = 1$, (5–3) is not satisfied, and it turns out that, for this special case, it is not possible to find a nonlinear Nash solution. The reason for this singularity in the Nash solution is that, when $\beta = 1$, the original game can be converted into an equivalent team problem (standard *LQ* optimal regulator problem) which is well-known to admit a unique linear solution. We note that, with $\beta \neq 1$ and $|x_0| \leqslant M$ for any arbitrary but fixed positive constant M, it is always possible to find a scalar p such that conditions (5–3) and (5–4) are satisfied. Hence, we have the following theorem.

Theorem 2.1. For the 2-person LQNZS game posed and with $\beta \neq 1$, $|x_0| \leqslant M$, there exist *nonlinear* Nash equilibrium solutions. One such solution set is given by

$$u^*(1) = -\tfrac{1}{2}x(1) + px^2(1) - p[x_0 + \gamma_0{}^*(x_0) + \gamma_2{}^*(x_0)]^2, \tag{6-1}$$

$$u^*(0) = \gamma_0{}^*(x_0), \tag{6-2}$$

$$v^*(0) = \gamma_2{}^*(x_0), \tag{6-3}$$

where $\gamma_0{}^*(\cdot)$, $\gamma_2{}^*(\cdot)$ are given by (5–1) and (5–2), respectively, and p is scalar satisfying

$$0 < p(1-\beta) \leqslant [(1+\beta)/4 + 3/2]^2/4M$$

or

$$-[(1+\beta)/4 + 3|2]^2/4M \leqslant p(1-\beta) < 0.$$

Proof. This is a direct consequence of the discussion given prior to the statement of Theorem 2.1. The Nash property of the triple (6) can also be verified directly by showing that it satisfies Inequalities (3). The conditions on p merely ensure that Ineq. (5–4) is satisfied.

3. Comparison of Different Nash Costs and Discussion

Let us now assume that $\beta = 0$ and determine the optimal Nash costs, corresponding to these nonlinear Nash strategies, as functions of the parameter p. Substituting (6) into (2), with $\beta = 0$, we have

$$J_1^*(p) = (49/16p^2)(1/8)\{-1 + \sqrt{[1 + (64p/49)\,x_0]}\}^2, \tag{7-1}$$

$$\begin{aligned} J_2^*(p) = {} & (1/16)(49/16p^2)\{-1 + \sqrt{[1 + (64p/49)\,x_0]}\}^2 \\ & + [(21/16p)\{-1 + \sqrt{[1 + (64p/49)\,x_0]}\} - x_0\}]^2. \end{aligned} \tag{7-2}$$

The closed-loop no-memory Nash solution, however, is given by (with $\beta = 0$) [this corresponds to the solution given in Refs. 1–3 for the continuous-time version of the problem]

$$u^*(1) = -\tfrac{1}{2}x(1), \qquad u^*(0) = -\tfrac{2}{7}x_0\,, \qquad v^* = -\tfrac{1}{7}x_0\,, \tag{8-1}$$

and the corresponding unique Nash costs under this no-memory assumption are

$$J_{1l}^* = 0.2495x_0^2, \tag{8-2}$$

$$J_{2l}^* = 0.102x_0^2. \tag{8-3}$$

Comparing (8) with (7) we observe that Player 1 does much better with a nonlinear solution (for large values of p) than he does with the linear solution given by (8–1), since $J_1^*(p) \to 0$ in the limit as $p \to \infty$. Referring back to (2–1), we note that the minimum possible value of J_1 is zero and, hence, Player 1 does the best that he can possibly do, with the nonlinear strategy (6–1) and (6–2) for large values of p. However, it can be shown that the nonlinear solution does not bring any advantage to Player 2 and he does worse with (6–3) than he would do with (8–1). Consequently, Player 2 would insist on sticking to a linear strategy, hence creating an ambiguous situation which necessitates communication of some kind between the two players, in order to arrive at an acceptable compromise.

It should be clear from the above that the method used to obtain the nonlinear Nash strategies is not limited only to the 2-stage problem considered in the paper, but can be used to obtain nonlinear solutions for multistage LQNZS games with the classical CL information structure for at least one of the players. This implies that equilibrium solutions of deterministic multiperson–multiobjective decision problems will not be unique in general, which is a serious threat to the validity of the existing results in the literature on game theory.

4. Appendix

Substituting (4–1) with $\psi(y, z) = (y^2 - z^2)p$ into J_1 and J_2, we have

$$J_1(u(0), v(0)) = [\tfrac{1}{2}x(1) + px^2(1) - \bar{x}_1^2 p]^2 + u^2(0) + [p\bar{x}_1^2 - px^2(1) + \tfrac{1}{2}x(1)]^2, \tag{9-1}$$

$$J_2(u(0), v(0)) = [\tfrac{1}{2}x(1) + px^2(1) - p\bar{x}_1^2]^2 + v^2(0) + \beta[\bar{x}_1^2 p - px^2(1) + \tfrac{1}{2}x(1)]^2. \tag{9-2}$$

Reaction curves of Players 1 and 2 can be found by solving, respectively,

$$\begin{aligned} dJ_1/du(0) = {} & [\tfrac{1}{2}x(1) + px^2(1) - \bar{x}_1^2 p][1 + 4px(1)] \\ & + 2u(0) + [\bar{x}_1^2 p - px^2(1) + \tfrac{1}{2}x(1)][1 - 4px(1)] = 0, \quad (10\text{-}1) \\ dJ_2/dv(0) = {} & [\tfrac{1}{2}x(1) + px^2(1) - p\bar{x}_1^2][1 + 4px(1)] \\ & + 2v(0) + \beta[\bar{x}_1^2 p - px^2(1) + \tfrac{1}{2}x(1)][1 - 4px(1)] = 0. \quad (10\text{-}2) \end{aligned}$$

Intersection points of these reaction curves determine the Nash strategies for the static game (9), as functions of γ_0 and γ_2. Now, if we want to obtain the Nash strategies to the original dynamic game, it will be sufficient to replace $u(0)$ by γ_0 and $v(0)$ by γ_2 in (10) and solve for common solutions to the resulting equations. In fact, making the substitutions in (10) will result in the simpler expressions

$$\bar{x}_1 + 2\gamma_0 = 0, \tag{11-1}$$

$$\tfrac{1}{2}\bar{x}_1(1 + \beta) + 2\bar{x}_1^2 p(1 - \beta) + 2\gamma_2 = 0, \tag{11-2}$$

where $\bar{x}_1$ is defined by (4–2).

A common solution to (11) is given by (5–1) and (5–2), and the conditions of intersection of the reaction curves are (5–3) and (5–4). It is not difficult now to see that (5–1), (5–2), and (6–1) satisfy Inequalities (3) and, hence, constitute a Nash equilibrium solution to the problem.

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